113 Class Problems: Ideals and Homormorphisms

1. Let $\mathbb{C}[x,y]$ be the ring of polynomials in two variables with complex coefficients. For example $2x^2y - 4x + 9y + 3 \in \mathbb{C}[x,y]$. The addition and multiplication are given by the usual addition and multiplication of polynomials. Prove that the following map is a homomorphism:

$$\begin{aligned} \phi : \mathbb{C}[x,y] &\to \mathbb{C} \\ f(x,y) &\mapsto f(0,0) \end{aligned}$$

Give an explicit description of the kernel.

Solutions:

Let $f(x,y), g(x,y) \in \mathbb{C}[x,y] \Rightarrow$ $f(x,y) = a_0 + f_1(x,y)x + f_2(x,y)y$, $g(x,y) = b_0 + g_1(x,y)x + g_2(x,y)y$

1/ $\phi(f(x,y) + g(x,y)) = \phi(a_0 + b_0 + (f_1(x,y) + g_1(x,y))x + (f_2(x,y) + g_2(x,y))y)$
$= a_0 + b_0 = \phi(f(x,y)) + \phi(g(x,y))$

2/ $\phi(f(x,y)g(x)) = \phi(a_0 b_0 + k(x,y)x + l(x,y)y) = a_0 b_0 = \phi(f(x,y))\phi(g(x,y))$

3/ $\phi(1) = 1$

$\operatorname{Ker}\phi = \{f(x,y) \mid f(0,0) = 0\} = \{f(x,y) \mid f \text{ has zero constant term}\}$
$= \{g(x,y)x + h(x,y)y \mid g,h \in \mathbb{C}[x,y]\}$

2. Let R be a ring and $I \subset R$ an ideal. We say I is a proper ideal if $I \neq R$. Prove the following:

I is a proper ideal of $R \iff I \cap R^* = \emptyset$.

Solutions:

Let's prove $I = R \iff I \cap R^* \neq \emptyset$

($\Rightarrow$)
$I = R \Rightarrow 1_R \in I \Rightarrow I \cap R^* \neq \emptyset$

($\Leftarrow$) Let $a \in I \cap R^* \Rightarrow 1_R = a^{-1}a \in I$

Let $r \in R \Rightarrow r = r \cdot 1_R \in I \Rightarrow I = R$

3. Let R be a commutative ring and $x \in R$. Define the subset

$$(x) := \{rx | r \in R\} \subset R$$

(a) Prove that (x) is an ideal. Any ideal of this form is called principal.

(b) Prove that (x) is proper if and only if $x \notin R^*$.

(c) (Hard) Give an example of a commutative ring R and an ideal I, such that I is not principal, i.e not of the form (x) for some $x \in R$. Carefully justify your answer.

Solutions:

a) 1/ $0_R \cdot x = 0_R \Rightarrow 0_R \in I$

2/ $rx + sx = (r+s)x \in I \quad \forall r, s \in R$

3/ $-(rx) = (-r)x \in I \quad \forall r \in R$

4/ $s(rx) = (sr)x \in I \quad \forall s, r \in R$

b) Claim $(x) = R \Leftrightarrow x \in R^*$

$(\Rightarrow)$ $(x) = R \Rightarrow \exists\, r \in R$ such that $rx = 1_R \Rightarrow x \in R^*$

$(\Leftarrow)$ $x \in R^* \Rightarrow \exists\, r \in R$ such that $rx = 1_R \Rightarrow 1_R \in (x) \Rightarrow (x) = R$

c) $\operatorname{Ker}\phi$ from Q1. $(f(x,y)) = \operatorname{Ker}\phi \Rightarrow f(x,y) | x$ and $f(x,y) | y$

$\Rightarrow f(x,y)$ constant

$\Rightarrow \operatorname{Ker}\phi = \mathbb{C}[x,y]$

Contradiction.

4. Let I, J be ideals in a commutative ring R.

(a) Prove that $I \cap J$ is an ideal.

(b) Let

$$I + J := \{x + y | x \in I,\ y \in J\}.$$

Prove that $I + J$ is an ideal.

Solutions:

a) • $0_R \in I, J \Rightarrow 0_R \in I \cap J$

• $x, y \in I \cap J \Rightarrow x + y \in I,\ x + y \in J \Rightarrow x + y \in I \cap J$

• $x \in I \cap J \Rightarrow -x \in I,\ -y \in J \Rightarrow -x \in I \cap J$

• $x \in I \cap J,\ r \in R \Rightarrow rx \in I,\ rx \in J \Rightarrow rx \in I \cap J$

b) • $0_R \in I,\ 0_R \in J \Rightarrow 0_R = 0_R + 0_R \in I + J$

• $(\underbrace{x_1}_{I} + \underbrace{y_1}_{J}) + (\underbrace{x_2}_{I} + \underbrace{y_2}_{J}) = (\underbrace{x_1 + x_2}_{I}) + (\underbrace{y_1 + y_2}_{J})$

• $-(\underbrace{x_1}_{I} + \underbrace{y_1}_{J}) = (\underbrace{-x_1}_{I}) + (\underbrace{-y_1}_{J})$

• $r(\underbrace{x_1}_{I} + \underbrace{y_1}_{J}) = (\underbrace{rx_1}_{I}) + (\underbrace{ry_1}_{J})$

$\forall\, x_1, x_2 \in I$
$y_1, y_2 \in J$
$r \in R$